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10. GAME THEORY USING FOR DECISION PROCESS MODELING

Keywords

Game Theory, Non-cooperative Games

Introduction

In contemporary economics the game theory is a popular modeling method. It can be used to describe behaviour characteristic for oligopolistic markets (including cartels, Cournot models, Stackelberg, Bertrand, Sweeze and many others). Using this theory one can model tenders' participants decisions, auctions, investments, decisions taken under risk or uncertainty. It finds application also in the theory of public choice to describe choices of citizens, social groups, political parties etc.

The origins of the game theory application in economics date back to the XIXth century. In 1838 a key publication was issued: 'Recherches sur les Principles Mathematiques de la Theorie des Richesses' written by Augustin Cournot, in which the author analysed the oligopolistic market using the non-cooperative game method. The following century brought about a dynamic expansion of the method – Emile Borel formulated a model of a zero-sum game with two participants, John von Neumann formulated models of sequential games which he developed and completed with other models together with Oskar Morgenstern in a book called 'Theory of Games and Economic Behavior' published in 1944 (Teoretyczne aspekty..., 2005, pp. 77-82). John Nash worked out an equilibrium concept, which was an alternative to the optimum of Vilfredo Pareto. For achievements in the field of game theories he was awarded the Nobel Prize together with John Harsang and Reinhard Selten (Malawski, Wieczorek and Sosnowska, 2004, p. 7).

Nowadays the game theory is an advanced branch of mathematics, however, still the simplest forms of games (such as payoff matrixes and game trees) serve well in explanations of many economic models being at the same time an effective method facilitating presentations of issues for teaching purposes (Ekonomia matematyczna..., 2006, pp. 93-96). In this article we are going to show some simple examples of economic situations that may be used in classes with students.

Formal notation of a game

The subject of the article is the **non-cooperative games** in which participants do not reach any agreement. A game will describe any situation that requires from an economic subject (consumer, entrepreneur, investor etc.) making a decision either without knowledge of other players' choices (simultaneous games – participants of the game make decisions parallely, due to which they do not know other participants' decisions) or in conditions of certainty – sequential games – subjects make decisions in a definite order knowing decisions of their antecedents.

If there are two participants in a game, the game may be noted as follows:

- Sets of strategies of individual players: $S_1 = \{s_1^1, s_2^1, \dots, s_m^1\}, S_2 = \{s_1^2, s_2^2, \dots, s_n^2\}$, m, n $\in$ N ,
- payoffs in *i*-th strategy of the first player and *j*-th strategy of the second player: $w_1(s_i^1, s_j^2) = a_{ij}$, oraz $w_2(s_j^2, s_i^1) = b_{ij}$, i = 1,2,..., m; j = 1,2,..., n;
- payoff matrixes accordingly: $A = \lfloor a_{ij} \rfloor_{mxn}$, $B = \lfloor b_{ij} \rfloor_{mxn}$,
- rules of the game: *R,*
- game: $G = \{S_1, S_2, A, B, R\}$.

Normal (matrix) form of a game and extensive (game tree) form of a game

Figure 1 presents a game in a normal form, which is payoff matrix. Numbers in the matrix cell indicate the payoffs, that are results which the players may achieve at a given combination of the participants' decisions. The payoff may be an economic result or other benefits e.g. units of utility. The first number in each cell is the payoff of the A player, the other number, payoff of the B player. Since the payoff matrix in fact contains two matrixes of each player, such games are referred to as double-matrix games (Malawski, Wieczorek and Sosnowska, 2004, p. 25).

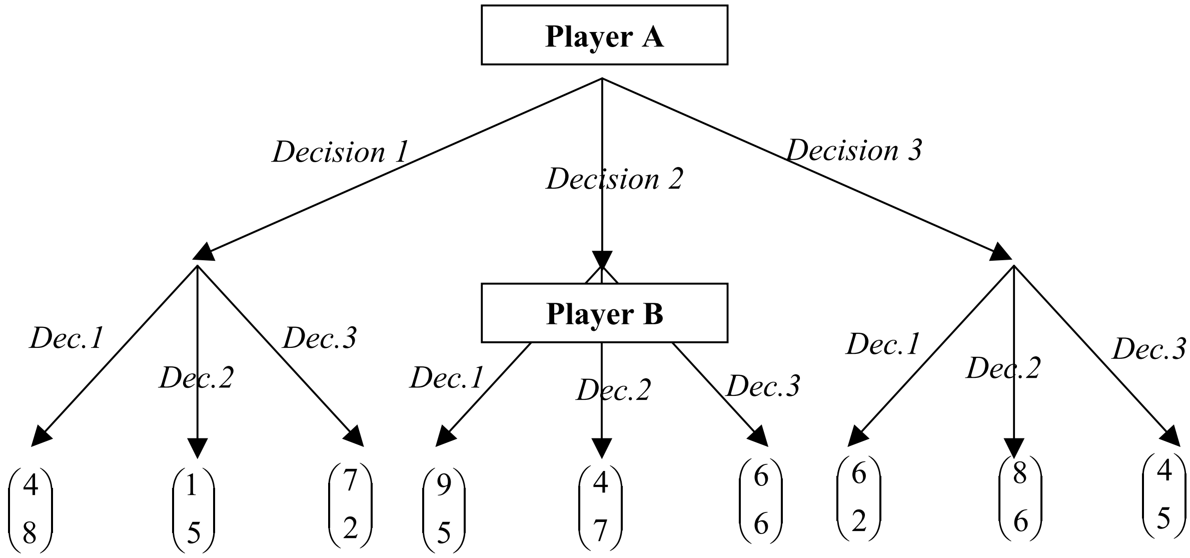
Figure 1. Example of a normal form game.

		Player B		
		Decision 1	Decision 2	Decision 3
Player A	Decision 1	4 8	1 5	7 2
	Decision 2	9 5	4 7	6 6
	Decision 3	6 2	8 6	4 5

Source: personal data.

Instead of payoff matrix, an extensive form of a game can be applied which is a game tree. Figure 2 presents a game tree for the game from the example shown in Fig.1. In more complicated games one departs from a graphic presentation for the sake of formalized mathematical notations (Ekonomia matematyczna..., 2006, pp. 93-96).

Figure 2. Extensive form of a game from Fig.1.



Source: personal data.

Dominant and dominated strategies and Nash equilibrium in simultaneous games.

In a **simultaneous game** it is assumed that each player makes an independent decision at the same time, so they do not know what the opponent will do. Each player knows, however, the opponent's potential strategies and payoffs possible to obtain in the game. The players aim at maximizing a payoff which is done after each player's turn (Kozubski, 1999, pp. 92-93).

The players, aiming at maximizing payoffs, make choices consisting in applying a particular strategy. If there is a dominant strategy in a game, it is the one that is the most profitable. A **dominant strategy** is the one that leads to best results regardless of the opponent's/opponents' decisions. However, if there is a strategy that never brings about the best outcome (no matter what the opponent will do), then it is referred to as a **dominated strategy** and it can be passed over in further analysis, since choosing it would not be rational (Ekonomia matematyczna..., 2006, p. 94).

In a simultaneous game it can be noted as follows – for the first player the strategy s_k^1 is a strategy dominated by the dominant strategy s_l^1, when the following occurs $a_{kj} \le a_{lj}, j = 1,2,...,n; k \ne l, k, l = 1,2,...,m.$ Analogically for the other player (Kozubski, 1999, p. 97).

If dominant strategies occur in a game, their choice leads to **Nash equilibrium**. It does not mean that in games deprived of dominant strategies such equilibrium does not exist. The Nash equilibrium is a situation where assuming a certain strategy of other players, whatever strategy change is done by a particular player, it leads to a worse payoff than the current strategy. Games may have more than one Nash equilibrium. If there are N players, then $s_* = \left(s_*^1, \ldots, s_*^N\right)$ is a Nash equilibrium if it satisfies the condition: $\forall s_j^i \in S_i$ $a\left(s_*^i, s_*^{-i}\right) \ge a\left(s_j^i, s_*^{-i}\right)$. The subscript – i denotes all the players except for the player i (Łyszkiewicz, 2000, p.110).

Case 1

Two travel agencies, Rex and Sphinx, want to extend their offer with an additional trip. Market research indicated that customers show equal interest in trips to three countries: A, B and C. The profits (in thousands PLN) from the inclusion of the additional trip to the offer are presented in the matrix below:

		Rex		
		Country A	Country B	Country C
Sphinx	Country A	51 57	50 62	61 60
	Country B	44 48	49 64	60 52
	Country C	46 51	40 52	51 50

Tasks:
- Check whether any company has a dominant strategy.
- Check whether any company has a dominated strategy.
- Point out the Nash equilibrium.

Solution of case 1

The dominant strategy of the travel agency Rex is extending the offer by a trip to country B, however, the remaining strategies are dominated strategies. The dominant strategy of the company Sphinx is extending the offer by a trip to country A, the dominated strategies are the trips to countries B and C. The Nash equilibrium occurs when Rex chooses country B and Sphinx chooses country A.

Case 2

Two companies, Garfield and Odie, want to enter the market with a new product. The applied marketing strategy will determine what market share of the target market they will win. They have to choose from two strategies – the first one entails carrying out an intensive media campaign, the other strategy demands promotions and distribution of free samples of the product. The companies have money to cover the costs of one strategy only; they have no possibility to finance both strategies at once, so they have to choose only one. The market share with different choices made by companies is shown in matrix below (data in percentages).

		Garfield	
		Strategy 1	Strategy 2
Odie	Strategy 1	60 40	30 70
	Strategy 2	50 50	60 40

Tasks:
- Check whether any company has a dominant strategy.
- Check whether any company has a dominated strategy.
- Point out the Nash equilibrium.

Solution of case 2

The companies do not have dominant strategies. They do not have dominated strategies either. None of the matrix cells is the Nash equilibrium.

Constant - sum games.

If in each situation (configuration of decisions) the sum of payoffs of all the players is constant (the same), then such a game is called a **constant-sum game**. According to the accepted notations a constant sum game for the case of two players may be defined formally: $\forall i, j \quad a_{ij} + b_{ij} = c, c - const$, so the sum of payoffs in each cell of the matrix must amount to the constant *c.* These games describe economic situations that come down to a division of a certain amount among the players (Kucharski, 2003, p. 9).

An exceptional case of a constant-sum game is a **zero-sum game**, also called an antagonistic game. A zero-sum game occurs when the constant *c* equals 0, so when $a_{ij} = -b_{ij}$. Wining of one player here means the loss of the other player. Since the payoff of one player defines at the same time the payoff of the other player, only one matrix of payoffs *A* will suffice to note such a game *(B = c - A).* Hence, the constant-sum games for two players are called matrix games (Malawski, Wieczorek and Sosnowska, 2004, p. 25).

Figure 3 presents a matrix for a zero-sum game in a normal form. The payoffs presented in the matrix are payoffs of player A, whereas player B in each situation reaches losses that equal the profits of player A.

Figure 3. Example of a zero-sum game in a normal form.

		Player B		
		Decision 1	Decision 2	Decision 3
Player A	Decision 1	7	6	8
	Decision 2	1	4	6
	Decision 3	9	5	2

Source: personal data.

Thanks to special qualities of a constant-sum game, the theory can provide a clear-cut solution (equilibrium). A von Neumann theorem called the **minimax theorem** states that in each matrix game there are optimal strategies of both players and precisely one value of a game. The value of a matrix game is its solution (minimax of a game), and the strategies leading to it, optimal strategies. In games of sums other than a constant one, the concept of an optimal strategy and value of a game usually does not make sense (Malawski, Wieczorek and Sosnowska, 2004, pp. 27-28).

In a case of a matrix game for two players, the solution of a game can be found in a set of pure strategies, if the following condition is met: $\max_i \min_j a_{ij} = \min_j \max_i a_{ij}$ (Gass, 1963, p. 275). The condition denotes the so-called 'saddle point'. It can be found by defining the lowest of the highest elements in a column and the highest of the lowest elements in a matrix row. Based on this condition we can find the equilibrium in the example shown in figure 3. The condition is met for a situation in which player A makes a decision 1 obtaining the winning amounting to 2, whereas player B makes decision 2 obtaining a payoff 6.

Not every game has equilibrium in the set of pure strategies. The **pure strategy** is in simplest words a strategy taken only once. If there are no strategies that satisfy the minimax condition then a randomization of a game takes place, which means looking for solutions in a set of mixed strategies. A **mixed strategy** is a linear convex combination of pure strategies which denotes the frequency of making a particular decision in a case of multiple choices. In a case of a single choice it can be the probability of a choice or the proportion of partial decisions (if the decision consists of partial decisions) (Badania operacyjne, 2001, p. 235).

To set about looking for a solution, elimination of dominated strategies occurs first, due to which the obtained matrix of payoffs A will be a square matrix. In further parts, only the players' strategies are analysed, those that remained after the dominated ones had been eliminated: $\{s_i^1\}_{i=1,...,k}$, $\{s_i^2\}_{i=1,...,k}$ [1]. Let vector $P=[p_1 \quad ... \quad p_k]$ denote the likelihood of a breakdown of the first player applying strategies, while vector $Q=[q_1 \quad ... \quad q_k]$ means the likelihood of the other player. Certainly the following conditions must be fulfilled: $\sum_{i=1}^{k} p_i = 1$, $\sum_{i=1}^{k} q_i = 1$. The mixed strategy of the first player will be the one as follow: $x = p_1 s_1^1 + p_2 s_2^1 + ... + p_k s_k^1$ and accordingly the mixed strategy of the other player: $y = q_1 s_1^2 + q_2 s_2^2 + ... + q_k s_k^2$.

To find mixed strategies of equilibrium x^*, y^*, one should solve two sets of equations. The first set (for the first player):

$$\begin{cases} P \circ \{a_{i1}\}_{i=1,...,k} = p_1 a_{11} + p_2 a_{21} + ... + p_k a_{k1} = v \\ ... \\ P \circ \{a_{ik}\}_{i=1,...,k} = p_1 a_{1k} + p_2 a_{2k} + ... + p_k a_{kk} = v \end{cases},$$

creates k equations with k unknowns: $p_1,..., p_{k-1}, v$, where $p_k = 1-(p_1 + ... + p_{k-1})$.

Analogically for the other player:

$$\begin{cases} Q \circ \{a_{1i}\}_{i=1,...,k} = q_1 a_{11} + q_2 a_{12} + ... + q_k a_{1k} = v \\ ... \\ Q \circ \{a_{ki}\}_{i=1,...,k} = q_1 a_{k1} + q_2 a_{k2} + ... + q_k a_{kk} = v \end{cases}.$$

Solution of case 2 – continuation

It was not possible to find the Nash equilibrium in the set of pure strategies through the strategy dominance procedure. It is easily noticed, however, that the game is a constant sum game – the sum of payoffs in each cell equals 100. It can be presented in the form of matrix containing only Odie's payoffs:

		Garfield	
		Strategy 1	Strategy 2
Odie	Strategy 1	60	30
	Strategy 2	50	60

Let's check that indeed the minimax condition is not satisfied here:
$\max_i \min_j a_{ij} = 50 \neq \min_j \max_i a_{ij} = 60$. According to the von Neumann theorem there is a solution in the set of mixed strategies. In the above example neither player has dominated strategies.

Each player has two strategies and thus: $P = [p, 1-p]$, $Q = [q, 1-q]$

[1] To simplify we assume that strategies have been indexed accordingly.

Set of equations for Odie:

$$\begin{cases} 60p + 50(1-p) = v \\ 30p + 60(1-p) = v \end{cases},$$

$10p + 50 = 60 - 30p$, and thus:

$p = \frac{1}{4}$ and $1 - p = \frac{3}{4}$.

For Garfield:

$$\begin{cases} 60q + 30(1-q) = v \\ 50q + 60(1-q) = v \end{cases},$$

$30q + 30 = 60 - 10q$, and thus:

$q = \frac{3}{4}$ and $1 - q = \frac{1}{4}$.

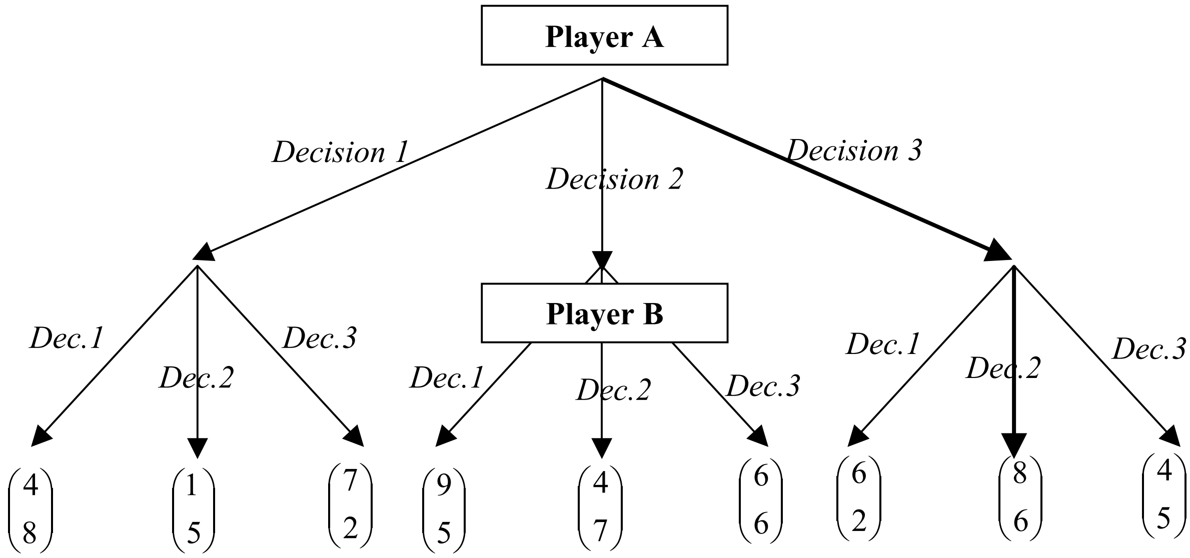
$x^* = 0{,}25 s_1^1 + 0{,}75 s_2^1$, $y^* = 0{,}75 s_1^2 + 0{,}25 s_2^2$.

It means that Garfield should apply the first strategy with a probability equal 0.75 (frequency 3/4), and the second strategy with a probability of 0.25 (frequency 1/4). Odie, the opposite: it should apply the first strategy with a probability of 0.25 (frequency 1/4), and the second strategy with a probability of 0.75 (frequency 3/4).

Sequential games

If in a game the decisions are not taken by players simultaneously, but in a pre-set order, then the game is referred to as a sequential game. The following decision-maker knows what strategy his rival chose, which changes the situation of the players – the subject making the decision earlier must predict what his opponent will do. Hence, the analysis of the game starts from the participant that has made a decision as the last one. Through elimination, the options that have not been chosen by a player are excluded. Such a process is a method of **backward induction**. The most convenient form to solve an uncomplicated case of a sequential game is a game tree (Ekonomia matematyczna, 2006, p. 94). Figure 4 shows such a tree with the equilibrium marked.

Figure 4. The Nash equilibrium in a sequential game.



Source: personal data.

An analysis of a tree in a sequential game starts from the bottom, so in the case presented in Fig.4, from player B. It can be noticed that the player will take the following decisions:

- Decision 1, if Player A chooses Decision 1 (it provides the highest payoff $8 > 5 > 2$);
- Decision 2, if Player A chooses Decision 2 ($7 > 6 > 5$),
- Decision 3, if Player A chooses Decision 3 ($6 > 5 > 2$).

Player A by predicting such moves of Player B takes into consideration only three payoffs: 4 if he chooses Decision 1, 4 with a choice of Decision 2, 8 with a choice of Decision 3. The highest payoff is 8, so finally he will choose Decision 3. Thus the equilibrium occurs with a choice of Decision 3 by Player A and Decision 2 by Player B.

Summary

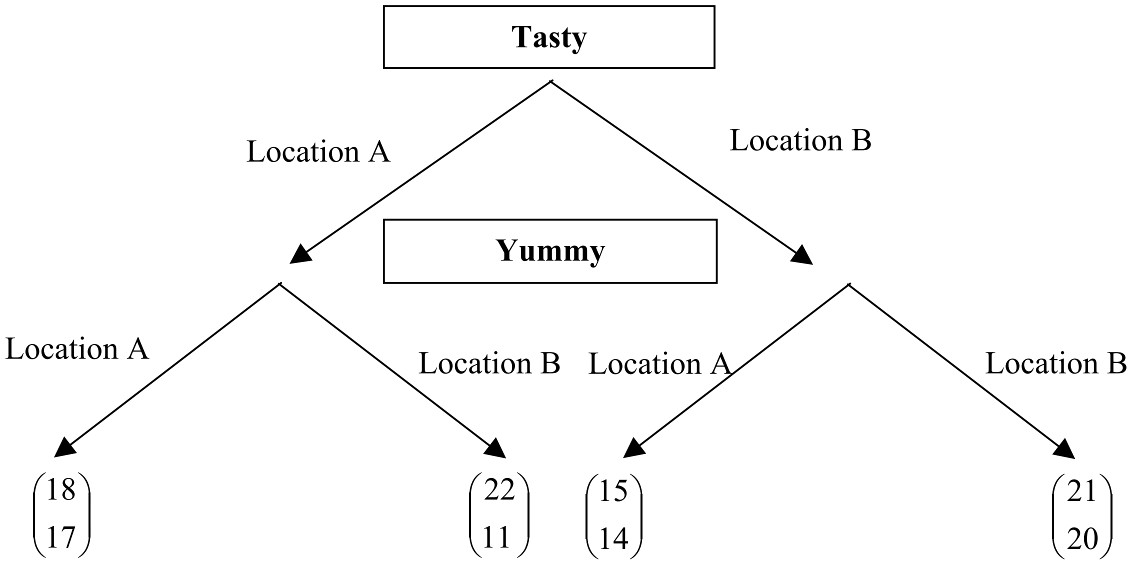
This work has discussed methods of modeling simple decision-making processes. It described how the dominance method may be used to solve, that is to find the Nash equilibrium, in simultaneous games in normal and extensive forms. It was also shown in what way overruling of simplifying assumptions leads to finding the Nash equilibrium in a set of mixed strategies. Finally it was also described how to model sequential decision-making. These issues are merely an introduction to the game theory; they just put an emphasis on the fact that this field opens to model decision-making processes.

Comprehension check

Case 4

In some city there are two companies: Tasty and Yummy. The Tasty company has a network of drink retail outlets, whereas the Yummy runs two small fast food bars. Both companies are planning to extend their activity through opening a new outlet in the City Centre. They have two locations to choose from: A and B.

The above situation is presented in a form of a game tree (the payoff in this case is the economic result of a company).



Tasks:

- The company Tasty is planning to open a new outlet on 14 Feb next year. The company Yummy will be granted a loan for activity expansion not earlier than in March next year. While making a decision about the outlet location, they will have already known Tasty's decision (sequential game). What decisions will the companies make?
- Present the problem in the form of a payoff matrix. What strategies will the companies choose if Yummy manages to obtain the money to open the new outlet earlier and the companies will make the decisions simultaneously (lack of both cooperation and knowledge of the opponent's decision)? Would then the companies have dominant strategies? Where the Nash equilibrium would be established?

Solution of case 4

The solution to a sequential game is a choice of a location B by both companies. If the companies made the decisions simultaneously, without cooperation, then the dominant strategy of Tasty would be location A. Yummy would not have a dominant strategy. The Nash equilibrium would be established in cell (20, 21).

		Tasty	
		Location A	Location B
Yummy	Location A	17 18	14 15
	Location B	11 22	20 21

Case 5

There are two colleges in some city: the public and the private one. Both of them plan to conduct an advertising campaign. The public collage has a good reputation and in experts' opinion advertising would not bring about any advantage. Collages compete for students from the same city, so the game is the 100-sum game. The percentage share of the market depends on advertisement expenditures (in thousands PLN) and is presented in a matrix below.

		Public Collage		
		0	**10**	**20**
Private Collage	**0**	50	20	50
	20	60	30	40
	40	20	70	70

Tasks:

- Find the Nash equilibrium in the simultaneous game.
- Find the Nash equilibrium in the sequential game. The private collage decides first, then the public one.

Solution of the case 5

First we check if the minimax condition is satisfied. The lowest element in the first row is 20, in the second one is 30 and in the third 20. Maximum of these values is 30. The highest element in the first column is 60, in the second one is 70 and in the third 70. Minimum of these is 60. $\max_i \min_j a_{ij} = 30 \neq \min_j \max_i a_{ij} = 60$. There is no solution in the set of pure strategies then.

We should eliminate dominated strategies before randomizing the game. Each of players has one dominated strategy: private collage – no advertisement (spending 0 PLN on advertisement) and public collage – spending 20 thousands PLN on advertisement. Reduced payoff matrix is presented below.

		Public Collage	
		0	**10**
Private Collage	**20**	60	30
	40	20	70

Set of equations for Private Collage:

$$\begin{cases} 60p + 20(1-p) = v \\ 30p + 70(1-p) = v \end{cases}.$$

$p = \frac{5}{8}$ and $1 - p = \frac{3}{8}$.

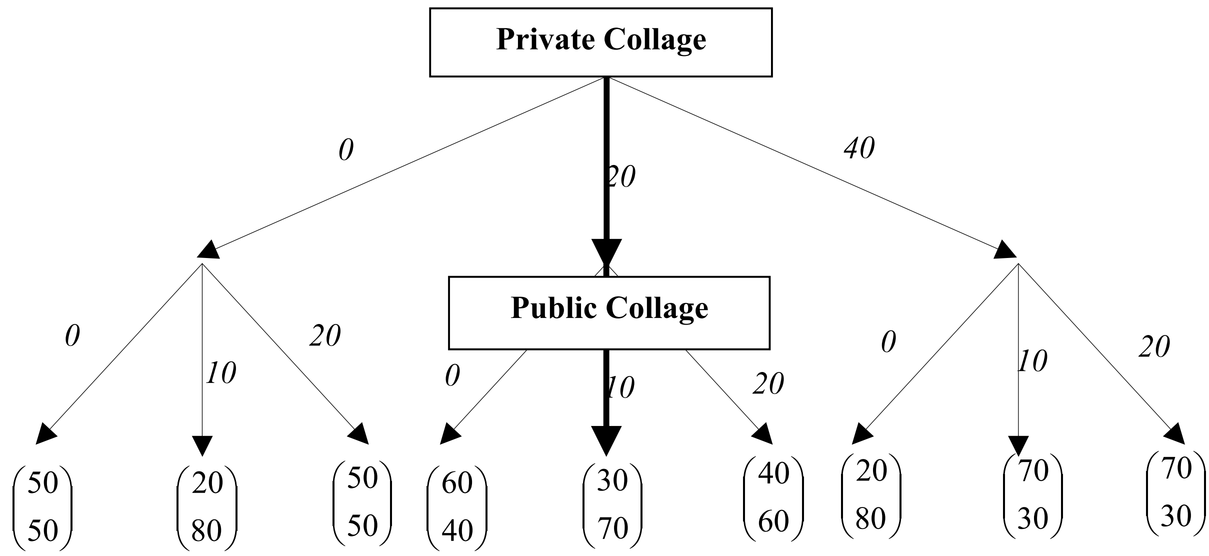
For Public Collage:

$$\begin{cases} 60q + 30(1-q) = v \\ 20q + 70(1-q) = v \end{cases}.$$

$q = \frac{1}{2}$ and $1 - q = \frac{1}{2}$.

There is an equilibrium when the private collage spends $\frac{5}{8} \cdot 20 + \frac{3}{8} \cdot 40 = 27{,}5$ thousands PLN on advertisement and the public collage spends $\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 10 = 5$ thousands PLN.

The solution of the sequential game is marked on the game tree below.



Questions for review

1. What is Nash equilibrium? Explain the difference between Nash equilibrium in a simultaneous and a sequence game.
2. What is pure and mixed strategy? Explain the interpretation of mixed strategy as a solution of the game.
3. Which form of game: normal or extensive is better to analyze? Discuss advantages and disadvantages of each form.

Recommended readings

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